Derivation of Least-Squares for $y = \beta_0 + \beta_1 x$

We seek to minimize the residual error R between the experimental values for the independent variable, y_i, and the values predicted by the model, $\hat{y}_i$

$$R = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

where n is the number of data points included in the model. First we replace $\hat{y}_i$ with $\hat{y} = b_0 + b_1 x$, where b_0 and b_1 are the estimates for β_0 and β_1

$$R = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

and then expand and simplify the terms within the summation

$$R = \sum_{i=1}^{n} (y_i^2 - 2b_0 y_i - 2b_1 x_i y_i + b_0^2 + 2b_0 b_1 x_i + b_i^2 x_i^2)$$

$$R = \sum_{i=1}^{n} y_i^2 - 2b_0 \sum_{i=1}^{n} y_i - 2b_1 \sum_{i=1}^{n} x_i y_i + nb_0^2 + 2b_0 b_1 \sum_{i=1}^{n} x_i + b_i^2 \sum_{i=1}^{n} x_i^2$$

Next, we take the partial derivative of R with respect to b_0 and set it equal to zero

$$\frac{\partial R}{\partial b_0} = -2\sum_{i=1}^{n} y_i + 2nb_0 + 2b_1 \sum_{i=1}^{n} x_i = 0$$

and solve for b_0

$$2nb_0 = 2\sum_{i=1}^{n} y_i - 2b_1 \sum_{i=1}^{n} x_i$$

$$\boxed{b_0 = \frac{\sum_{i=1}^{n} y_i - b_1 \sum_{i=1}^{n} x_i}{n}}$$

Then, we take the partial derivative of R with respect to b_1 and set it equal to zero

$$\frac{\partial R}{\partial b_1} = -2\sum_{i=1}^{n} x_i y_i + 2b_0 \sum_{i=1}^{n} x_i + 2b_1 \sum_{i=1}^{n} x_i^2 = 0$$

and solve for b_1

$$\sum_{i=1}^{n} x_i y_i - b_0 \sum_{i=1}^{n} x_i - b_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} x_i y_i - \left(\frac{\sum_{i=1}^{n} y_i - b_1 \sum_{i=1}^{n} x_i}{n} \right) \sum_{i=1}^{n} x_i - b_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$n \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} y_i - b_1 \sum_{i=1}^{n} x_i \right) \sum_{i=1}^{n} x_i - n b_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i + b_1 \left(\sum_{i=1}^{n} x_i \right)^2 - n b_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$n b_1 \sum_{i=1}^{n} x_i^2 - b_1 \left(\sum_{i=1}^{n} x_i \right)^2 = n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i$$

$$b_1 \left(n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2 \right) = n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i$$

$$\boxed{b_1 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2}}$$

The value of b_1 is calculated first and then used to calculate b_0.